

Jaynes principle *versus* entanglement

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We show, by explicit examples, that the Jaynes inference scheme based on maximization of entropy can produce *inseparable* states even if there exists a *separable* state compatible with the measured data. It can lead to problems with processing of entanglement. The difficulty vanishes when one uses inference scheme based on minimization of entanglement.

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It is well known that quantum mechanics allows to reconstruct completely a state of the quantum mechanical system from mean values of complete system of observables measured on the ensemble of identically prepared systems [1]. By complete set of observables [1] one means the maximal set of linearly independent observables where the trivial observable represented by identity operator is excluded. In practice we often deal with situations when the state of the system is unknown and only mean values a_i , ($i = 1, \dots, p$) of some *incomplete* set of observables $\{A_i\}_{i=1}^p$ are available from experiments i.e.

$$\text{Tr} \varrho A_i \equiv \langle A_i \rangle = a_i \quad i = 1, \dots, p. \quad (1)$$

Then of course there can be many states which are in agreement with the measured data. It involves the problem of estimation of the state on the basis of the exact mean values of given observables. According to the maximum entropy principle [2–4] we have to choose from a set of states ϱ which fulfil the constraint (1) the most probable (or representative) state ϱ_J which maximizes the von Neumann entropy

$$S(\varrho) = -\text{Tr} \varrho \ln \varrho. \quad (2)$$

Then the representative state ϱ_J is given by [3]

$$\varrho_J = Z(\boldsymbol{\lambda})^{-1} e^{-\sum_{i=1}^p \lambda_i A_i}, \quad (3)$$

where $Z(\boldsymbol{\lambda}) = \text{Tr} e^{-\sum_{i=1}^p \lambda_i A_i}$ is the partition function and the vector $\boldsymbol{\lambda}(a_1, \dots, a_p)$ is uniquely determined by the vector $\mathbf{a} = (a_1, \dots, a_p)$

$$-\frac{\partial \ln Z(\boldsymbol{\lambda})}{\partial \lambda_i} = a_i, \quad i = 1, \dots, p. \quad (4)$$

The above maximum-entropy principle (or Jaynes principle) was applied for partial reconstruction of pure and mixed states of many different systems [5]. In particular it allowed to interpret quantum statistical mechanics as a special type of statistical inference [3] based on the entropic criterion.

The Jaynes principle is the most rational inference scheme in the sense that it does not permit to draw any conclusions unwarranted by the experimental data. However, this argument making the principle plausible does not actually prove it [6]. In this context one can ask the question: is the entropic criterion universal? Surprisingly, as we will show in this paper there *are* situations where the Jaynes principle fails. This concerns compound quantum systems, which have recently attracted much attention due to the new phenomena such as quantum teleportation [7], quantum dense coding [9] or quantum cryptography [8].

In all the above effects the most important characteristics of state is entanglement (or inseparability) [10]. Suppose we need the entanglement to deal with one of these effects, having, however, the compound system in unknown state and some *incomplete* data of type (1). Then, usually, to proceed further, we must somehow estimate the state of the system from the data. But what scheme of inferring can be used in this case? The fact that we need the entanglement for our purposes imposes a basic condition on possible inference schemes. Namely they certainly should not give us inseparable estimated state if only theoretically there exists a separable state compatible with the measured data. Otherwise it may happen that we get into troubles trying to use the entanglement we inferred to be present, when

in fact there is no entanglement at all! Further it will be shown that the entropic criterion does not protect us from such situations.

Now, a basic question arises: how to check whether the given constraints could (could not) be satisfied by a separable state? A natural way is performing *minimization of entanglement*. Clearly the latter must be somehow quantified. To this end one uses the so-called measures of entanglement which vanish for separable states (the latter represent no entanglement) [12]. Hence a reasonable inference scheme should involve minimization of a chosen measure of entanglement. Then one can be sure that if the data could be produced by a separable state then the estimated state would also be separable.

In this paper we propose the inference scheme based on the entanglement criterion. The crux of the scheme is minimization of entanglement. This procedure is followed by maximization of entropy which ensures uniqueness of the resulting state. We present explicit examples of data for which the minimum entanglement state is *separable*, while the state obtained by means of Jaynes principle is *inseparable*. This occurs even if the mean values come from measurements made by distant observers who can only exchange classical information.

Let us make some remarks concerning the procedure of minimization of entanglement. Note first that since the entanglement measures vanish for separable states then they cannot be strictly convex state functions. As a consequence, under a given set of constraints of type (1) the state of minimum entanglement does not need to be unique. To overcome the difficulty, we propose to maximize the von Numann entropy *after* minimization of entanglement. Such a procedure produces *unique representative state* [13]. We shall denote it by ϱ_E where E is the used measure of entanglement.

In our analysis we will use two measures: entanglement of formation E_f [14] and relative entropy entanglement E_r [15]. Both of them are calculated for the two spin- $\frac{1}{2}$ states diagonal in the Bell basis [16] given by [17]

$$\begin{aligned}\psi_{(2)}^{\mp} &\equiv \Phi^{\mp} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \mp |\downarrow\downarrow\rangle) \\ \psi_{(0)}^{\pm} &\equiv \Psi^{\pm} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle).\end{aligned}\tag{5}$$

In this case both the measures depend only on the largest eigenvalue F of a given state and are increasing functions of F [14,15]

$$E_f = H\left(\frac{1}{2} + \sqrt{F(1-F)}\right), \quad E_r = \ln 2 - H(F)\tag{6}$$

for $F > \frac{1}{2}$ and $E_r = E_f = 0$ otherwise; here $H(x) = -x \ln x - (1-x) \ln(1-x)$. Therefore if the state of minimum entanglement is diagonal in the Bell basis it is of the *same* form for both measures (i.e. $\varrho_{E_f} = \varrho_{E_r}$).

Here we shall illustrate a fundamental difference between the entropic criterion and entanglement criterion of inference by means of examples involving constraints which are invariant under measurement in the Bell basis. They are characterized by the following condition: any state which fulfils the constraints would also satisfy them if subjected to measurement in the Bell basis. Such constraints will be further called *Bell constraints*. It turns out that for the Bell constraints the number of the state parameters which are to be varied within the minimization procedure can be considerably reduced. This follows from the following lemma.

Lemma.- *For the Bell constraints the representative state ϱ_E (where $E = E_f$ or $E = E_r$) is diagonal in the Bell basis.*

Proof.- To prove the lemma we note two important properties of the measurement in the Bell basis (5). Namely, such a measurement (i) does not increase entanglement (for the considered measures), (ii) does not decrease entropy. In other words for any state ϱ we have

$$E(\varrho_B) \leq E(\varrho), \quad S(\varrho_B) \geq S(\varrho)\tag{7}$$

where ϱ_B is the state resulting from ϱ after performing measurement in Bell basis

$$\varrho_B = \sum_{i=0}^3 P_i^B \varrho P_i^B\tag{8}$$

with $P_i^B = |\psi_i\rangle\langle\psi_i|$. The first property we prove in Appendix. The second one follows from the fact that the entropy does not decrease under von Neumann measurement [6] (one says that measurement *enhances mixing*).

Let us take the state ϱ_E which is representative under some Bell constraints. Consider a new state ϱ_E^B given by

$$\varrho_E^B = \sum_i P_i^B \varrho_E P_i^B.\tag{9}$$

By definition of the Bell constraints ϱ_E^B also satisfies them. According to the properties (i) and (ii) we have $E(\varrho_E^B) \leq E(\varrho_E)$ and $S(\varrho_E^B) \geq S(\varrho_E)$. But as ϱ_E is the representative state, then no other state satisfying the constraints can be less entangled, hence $E(\varrho_E^B) = E(\varrho_E)$. As the state ϱ_E is unique then among the states of minimum entanglement no other state can have entropy greater than or equal to ϱ_E . In result we have $\varrho_E^B = \varrho_E$. But this means that the state ϱ_E does not change under the measurement in Bell basis. This is possible if and only if ϱ_E is *diagonal* in this basis. This ends the proof of the lemma.

From the lemma it follows that for the Bell constraints one can perform the procedure of minimization of entanglement (and subsequent maximization of entropy) *only* over the Bell diagonal states and in this way would obtain the same result as if the procedure were performed over the *whole* set of states satisfying the constraints. Then, by formulas (6) the minimization of entanglement reduces for both measures to the minimization of the largest eigenvalue of the states, producing then the same representative state ϱ_E . In particular the formulas imply that the latter is inseparable if and only if the eigenvalue is greater than $\frac{1}{2}$.

Now we are in position to illustrate the difference between the two inference schemes. First consider the Bell-CHSH observable [18] $B = \sqrt{2}(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z) = 2\sqrt{2}(|\Phi^+\rangle\langle\Phi^+| - |\Psi^-\rangle\langle\Psi^-|)$ with the mean value

$$\text{Tr} \varrho B = b, \quad 0 \leq b \leq 2\sqrt{2} \quad (10)$$

(i.e. we have only one constraint). Let us first find the representative state ϱ_E . Of course, as B is diagonal in Bell basis, then it forms Bell constraints. Indeed for any state ϱ we have

$$b = \text{Tr} \varrho B = \text{Tr} \left(\varrho \sum_i P_i^B B P_i^B \right) = \text{Tr} \left(\sum_i P_i^B \varrho P_i^B B \right). \quad (11)$$

Hence the state after measurement still satisfies the constraints. Then we need to minimize the largest eigenvalue of the state of the form

$$\varrho = p_1 |\Phi^+\rangle\langle\Phi^+| + p_2 |\Psi^-\rangle\langle\Psi^-| + p_3 |\Psi^+\rangle\langle\Psi^+| + p_4 |\Phi^-\rangle\langle\Phi^-|, \quad (12)$$

where $\sum_i p_i = 1$, $p_i \geq 0$ and $p_1 - p_2 = \frac{b}{2\sqrt{2}}$. Note that if $b \leq \sqrt{2}$ then for $p_2 = \frac{1}{2} - \frac{b}{2\sqrt{2}}$ the state is separable as then the largest eigenvalue is $p_1 = \frac{1}{2}$ (we will not calculate the state ϱ_E in this case). For $b > \sqrt{2}$ the state (12) is always inseparable as $p_1 > \frac{1}{2}$. The latter is minimal if $p_2 = 0$. Then we obtain the family of states with minimal entanglement of the form

$$\varrho = \frac{b}{2\sqrt{2}} |\Phi^+\rangle\langle\Phi^+| + p_3 |\Psi^+\rangle\langle\Psi^+| + p_4 |\Phi^-\rangle\langle\Phi^-|, \quad (13)$$

Subsequently, maximizing the von Neumann entropy we obtain the representative state ϱ_E of the form

$$\varrho_E = \frac{b}{2\sqrt{2}} |\Phi^+\rangle\langle\Phi^+| + \left(\frac{1}{2} - \frac{b}{2\sqrt{2}} \right) (|\Psi^+\rangle\langle\Psi^+| + |\Phi^-\rangle\langle\Phi^-|), \text{ for } b > \sqrt{2}. \quad (14)$$

Thus we have shown that under the constraint (10) the representative state ϱ_E is separable for $b \leq \sqrt{2}$ and inseparable for $\sqrt{2} < b \leq 2\sqrt{2}$.

Let us now apply the Jaynes inference scheme to the same data. Then the Jaynes state, calculated directly by use of the formula (3) is given by

$$\begin{aligned} \varrho_J = \frac{1}{4} \left[\left(1 - \frac{b}{\sqrt{2}} + \frac{b^2}{8} \right) |\Phi^+\rangle\langle\Phi^+| + \right. \\ \left. \left(1 + \frac{b}{\sqrt{2}} + \frac{b^2}{8} \right) |\Psi^-\rangle\langle\Psi^-| + \left(1 - \frac{b^2}{8} \right) (|\Psi^+\rangle\langle\Psi^+| + |\Phi^-\rangle\langle\Phi^-|) \right]. \end{aligned} \quad (15)$$

The above state is inseparable for $b > 4 - 2\sqrt{2}$. Then in the range $4 - 2\sqrt{2} < b \leq \sqrt{2}$ the Jaynes inference produces *inseparable* state while the minimum entanglement state is *separable*. Although we used some particular entanglement measures, the result is *general*: the property that the state ϱ_E is separable (inseparable) does not depend on the type of measure E .

One could think that this difference between the two types of inference is due the fact that the used observable is nonlocal, i.e. it cannot be measured itself without interchange of quantum information between the observers. If the measurements are performed locally then the mean value of Bell-CHSH observable is not the only measured quantity as we simultaneously obtain the mean values of the product observables which add up to the observable. Moreover,

by measuring the product observable we gain an additional information. Indeed, if the correlations are measured, the marginal distributions are also obtained. Then consider the following data which could be obtained by distant observers (who can communicate only by means of classical bits)

$$\begin{aligned}\langle \sqrt{2}\sigma_x \otimes \sigma_x \rangle &= \langle \sqrt{2}\sigma_z \otimes \sigma_z \rangle = \frac{b}{2}, \\ \langle \sigma_x \otimes I \rangle &= \langle \sigma_z \otimes I \rangle = \langle I \otimes \sigma_x \rangle = \langle I \otimes \sigma_z \rangle = 0.\end{aligned}\tag{16}$$

One can check that these are again Bell constraints. Then the state ϱ_E can be derived similarly as in the previous case. Here we obtain the *same* results as in the case of the Bell-CHSH observable measured alone (except that for $b \leq \sqrt{2}$ the state ϱ_E may be of different form still however being separable).

Finally it is interesting to consider the projector P_0 corresponding to the singlet state vector Ψ^- , which is a manifestly nonlocal observable. One can check that here $\varrho_E = \varrho_J$ for any mean value $F = \text{Tr} \varrho P_0$ and both the states are equal to a suitable Werner state [11,20]. This involves an interesting problem: for which type of constraints the Jaynes scheme fails? However, it goes beyond the scope of this paper.

It should be mentioned here that the problem of the entanglement processing with incomplete data appeared implicitly in the context of the protocols of entanglement distillation. Indeed, the first proposed distillation scheme [21] is based on information about the state given just by the projector P_0 . In result, in contrast with more general schemes which involve full knowledge about the state [22], it works only for $F > \frac{1}{2}$. In the present context, this appears to be a consequence of the fact that the minimum entanglement state for $F \leq \frac{1}{2}$ is separable. Now if the real state is in fact inseparable, we must gain some more information (i.e. to increase the number of observables) to be able to distill the state.

Finally, one can ask what is the place of the two inference schemes (the entropic one and the entanglement one) in quantum communication theory. It seems that they are in a way complementary. As the quantum noisy channels are usually described in terms of entanglement [14,23], the scheme proposed in this paper could be a suitable tool for estimation of parameters of quantum channels. On the other hand, the capacity of a quantum source is described by von Neumann entropy [24]. Thus the Jaynes principle is here the natural scheme.

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APPENDIX A:

We will show here that $E(\varrho_B) \leq E(\varrho)$ where $\varrho_B = \sum_i P_i^B \varrho P_i^B$, $E = E_f$ or E_r .

To prove the above, note first that ϱ_B is diagonal in the Bell basis i.e. it is of the form $\varrho_B = \sum_i \lambda_i P_i^B$. Let λ_k be the largest eigenvalue. Consider the state $\varrho'_B = I \otimes \sigma_k \varrho_B I \otimes \sigma_k$ (where $\sigma_{1,2,3}$ are Pauli matrices, $\sigma_0 = I$) which is still diagonal in the Bell basis. Clearly we have $\varrho'_B = \sum_i P_i^B \varrho' P_i^B$, where $\varrho' = I \otimes \sigma_k \varrho I \otimes \sigma_k$. As we have $I \otimes \sigma_k P_k I \otimes \sigma_k = P_0$, the largest eigenvalue of ϱ'_B i.e. λ_k corresponds now to the singlet projector P_0 . Now, we will use the “twirling” operation [21] i.e. random unitary transformation of the form $U \otimes U$. The twirling converts any state $\tilde{\varrho}$ into the Werner state [11,21] $\varrho_W = \text{Twirl}(\tilde{\varrho})$ given by

$$\varrho_W = F P_0 + \left(\frac{1-F}{3}\right)(P_1 + P_2 + P_3),\tag{A1}$$

where $F = \text{Tr} \tilde{\varrho} P_0$. Now the equality $\text{Tr} \varrho'_B P_0 = \text{Tr} \varrho' P_0$ implies $\text{Twirl}(\varrho') = \text{Twirl}(\varrho'_B)$. Moreover, since for Bell diagonal state both entanglement measures depend only on the largest eigenvalue, we get $E(\text{Twirl}(\varrho'_B)) = E(\varrho'_B)$. Subsequently, as twirling involves only local quantum operations and classical communication we have $E(\text{Twirl}(\varrho')) \leq E(\varrho')$. Finally we obtain

$$E(\varrho'_B) = E(\text{Twirl}(\varrho'_B)) = E(\text{Twirl}(\varrho')) \leq E(\varrho')\tag{A2}$$

As entanglement measures are invariant under product unitary transformation (in particular under $I \otimes \sigma_k$ ones) we have also $E(\varrho_B) \leq E(\varrho)$.

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